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Waves in an electrically conducting rotating liquid

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The waves generated by the steady motion of an obstacle along the axis of a uniformly rotating, electrically conducting homogeneous fluid have been studied by Lighthill's technique. The wave-number surface consists of a sphere and four coincident planes. The waves corresponding to points on the sphere travel ahead or trail behind the obstacle according as a_1 the Alfvén velocity, is greater or less than U, the velocity of the obstacle. By drawing the appropriate normals to the four planes, it is seen that the formation of a Taylor column ahead of the obstacle is possible even at large Rossby numbers when $U < a_1$ in contrast with the non-magnetic case and the case with $U > a_1$.

1. Introduction

It has long been established that disturbances in uniformly rotating incompressible liquid can propagate as wave motions. Taylor (1922) has shown that rotating liquid can transmit plane and spherical waves of length $\pi U/\Omega$, where Ω is the angular velocity with which the liquid is rotating and U is the velocity of translation of a non-vibrating source of disturbance along the axis of rotation. A general theory with an application to rotating liquids has been developed by Lighthill (1967) to study the dispersive waves generated by travelling forcing effects which may be steady or oscillatory or transient in character. In this note Lighthill's theory has been applied to study the waves created by the steady motion of an obstacle along the axis of an electrically conducting homogeneous rotating liquid. The undisturbed motion consists of a rigid rotation Ω about an axis and a uniform magnetic field H_0 applied along the axis of rotation. Three cases arise according as $a_1 \gtreqless U$. When $U \gtrless a_1$, the wave-number surface consists of a sphere and four coincident planes and for $U = a_1$ it is just four coincident planes. The waves generated in the case $U > a_1$ are qualitatively similar to those without magnetic field. In the present case also the spherical waves trail behind the obstacle but their wavelengths are reduced by a factor $1-(a_1^2/U^2)$. The unattenuated waves, which are two-dimensional in character, propagate both ahead and behind the obstacle in a Taylor column and the waves ahead are subjected to a 'low-pass filter' passing only wave-numbers below $2\Omega U/(U^2-a_1^2)$. The wave-number range admissible in this case is bigger than the corresponding range in the non-magnetic case and hence the disturbances ahead in the Taylor column are intensified at small Rossby numbers. In contrast with the above case, for $U < a_1$, the spherical waves are found ahead of the obstacle and the unattenuated disturbances in the Taylor column are not subjected to any constraints.

2. The equations of motion

The equations, referred to a rotating frame of reference, governing the motion of an incompressible, inviscid, unbounded, infinitely conducting liquid rotating about Ox as a rigid body in the presence of an externally applied magnetic field are

$$\partial \mathbf{V}/\partial t + (\mathbf{V}.\nabla)\mathbf{V} + \mu/4\pi\rho \mathbf{H}\times\nabla\times\mathbf{H} + 2\Omega\hat{\imath}\times\mathbf{V} = \nabla p + \mathbf{g}_1(x-Ut, y, z), \tag{1}$$

$$\partial \mathbf{H}/\partial t - \nabla\times(\mathbf{V}\times\mathbf{H}) = 0, \tag{2}$$

$$\nabla.\mathbf{V} = 0, \quad \nabla.\mathbf{H} = 0, \tag{3}$$

where $\mathbf{V}$ is the velocity vector, $\mathbf{H}$ is the total magnetic field, μ is the permeability of the medium, ρ is the density, Ω is the angular velocity of the liquid, $\hat{\imath}$ the unit vector in x direction, $\mathbf{g}_1$ is a steady forcing term moving with velocity U in x direction and $p = (-p'/\rho) + \frac{1}{2}\Omega^2(y^2+z^2)$. In these equations the displacement current has been neglected ($\mathbf{V} \ll C$ the velocity of light).

The equations (1) and (2) on linearization, for a magnetic field H_0 applied in the x direction, reduce to

$$\frac{\partial \mathbf{V}}{\partial t} - \mu\frac{H_0}{4\pi\rho}\frac{\partial \mathbf{h}}{\partial x} + 2\Omega\hat{\imath}\times\mathbf{V} = \nabla p + \mathbf{g}_1, \tag{4}$$

$$\frac{\partial \mathbf{h}}{\partial t} - H_0\frac{\partial \mathbf{V}}{\partial x} = 0, \tag{5}$$

where $\mathbf{h}$ is the perturbed magnetic field. By taking $\partial/\partial t$ curl of (4) and making use of (5) and (3), we get

$$\left(\frac{\partial^2}{\partial t^2} - a_1^2\frac{\partial^2}{\partial x^2}\right)\operatorname{curl}\mathbf{V} - 2\Omega\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{V}}{\partial t}\right) = \frac{\partial \mathbf{g}}{\partial t}, \tag{6}$$

where $a_1^2 = H_0^2\mu/4\pi\rho$ and $\operatorname{curl}\mathbf{g}_1 = \mathbf{g}$. Again operating with $(\partial^2/\partial t^2 - a_1^2\,\partial^2/\partial x^2)\operatorname{curl}$ on (6) and after some manipulation, one finds†

$$\left(\frac{\partial^2}{\partial t^2} - a_1^2\frac{\partial^2}{\partial x^2}\right)^2\nabla^2\mathbf{V} + 4\Omega^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \mathbf{V}}{\partial x^2}\right) = \mathbf{f}(x-Ut, y, z), \tag{7}$$

where
$$\mathbf{f} = -\left(\frac{\partial^3}{\partial t^3}\operatorname{curl} - a_1^2\frac{\partial^3}{\partial x^2\,\partial t}\operatorname{curl} + 2\Omega\frac{\partial^3}{\partial x\,\partial t^2}\right)\mathbf{g}. \tag{8}$$

By taking three-fold Fourier transform, the formal solution of (7) is given by

$$\mathbf{V} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\mathbf{k})\exp[i\mathbf{k}.(\mathbf{r}-\hat{\imath}Ut)]}{P(Ul, l, m, n)}\,dl\,dm\,dn, \tag{9}$$

where $\mathbf{k} = l\hat{\imath} + m\hat{\jmath} + n\hat{\mathbf{k}}$ is the wave-number vector, $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}}$ is the position vector,

$$P(Ul, l, m, n) = (U^2l^2 - a_1^2 l^2)^2(l^2+m^2+n^2) - 4\Omega^2(U^2l^2)\,l^2, \tag{10}$$

† For a mass source disturbance, the forcing term $\mathbf{g}_1$ appears in the first of (3) instead of (1). In this case the equation (7) remains the same except for a change in right-hand side.

and $$\mathbf{f}(\mathbf{r}-\hat{\mathbf{\imath}}Ut) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k})\exp\left[i\mathbf{k}.(\mathbf{r}-iUt)\right] dl\,dm\,dn, \tag{11}$$

where $F(\mathbf{k}) = F(l,m,n)$ is a regular function for all l,m,n. The contributions to the integral (9) come from the roots of the equation

$$P(Ul,l,m,n) = 0, \tag{12}$$

which is called the wave-number surface and it consists of a sphere and four coincident planes. At each point of the wave-number surface, we draw an arrow normal to the surface choosing from the two normal directions the one pointing towards the surface

$$P(Ul+\delta,l,m,n) = 0, \tag{13}$$

with δ small and positive. Then the waves are found in the direction of the arrows at each point of the wave-number surface, stretching out from the forcing region. The amplitude of the waves corresponding to the sphere is asymptotically given by (Lighthill 1967)

$$\frac{4\pi^2 F(\mathbf{k})}{|K|^{\frac{1}{2}} R\,|\nabla P(Ul,l,m,n)|}, \tag{14}$$

where $R = |\mathbf{r}-\hat{\mathbf{\imath}}Ut|$ is the distance from the forcing region, ∇ is the operator grad with respect to l, m, n and K is the Gaussian curvature of the wave-number surface. The equation (14) cannot be used for the waves corresponding to multiply covered plane part of the wave-number surface. A modified method is given at the end of § 3.

Nigam & Nigam (1963) applied the methods of Lighthill (1960) to the more general problem of waves created by a periodic point source with frequency ω, placed at the origin of an unbounded electrically conducting rotating liquid with uniform velocity of translation U along the axis of rotation.

3. Discussion of the results

The wave-number surface is given by

$$l^4[(U^2-a_1^2)^2(l^2+m^2+n^2)-4\Omega^2U^2] = 0. \tag{15}$$

Three cases will arise according to whether $U >$ or $=$ or $< a_1$.

Case 1. $U > a_1$

The wave-number surface consists of a sphere

$$l^2+m^2+n^2 = \left(\frac{2\Omega U}{U^2-a_1^2}\right)^2 \tag{16}$$

and four coincident planes (figure 1). The wave-numbers on this sphere correspond to waves of uniform length $\pi(U^2-a_1^2)/\Omega U$ and arbitrary direction. The wavelength is reduced in comparison with the non-conducting liquid case for which it is $\pi U/\Omega$. The directions of the arrows on the sphere are such that these waves are found only behind the forcing region.

The surface (15) includes a straight portion, the plane $l = 0$ taken four times. The arrows along the appropriate normal must be drawn on the four planes and the normal directions appropriate to each plane may or may not coincide. By drawing the surface (13) for small positive δ, we observe that the plane $l = 0$ of (15) splits into four sheets, three of which lie below and one above $l = 0$ when

$$(m^2+n^2)^{\frac{1}{2}} < 2\Omega U/(U^2-a_1^2), \tag{17}$$

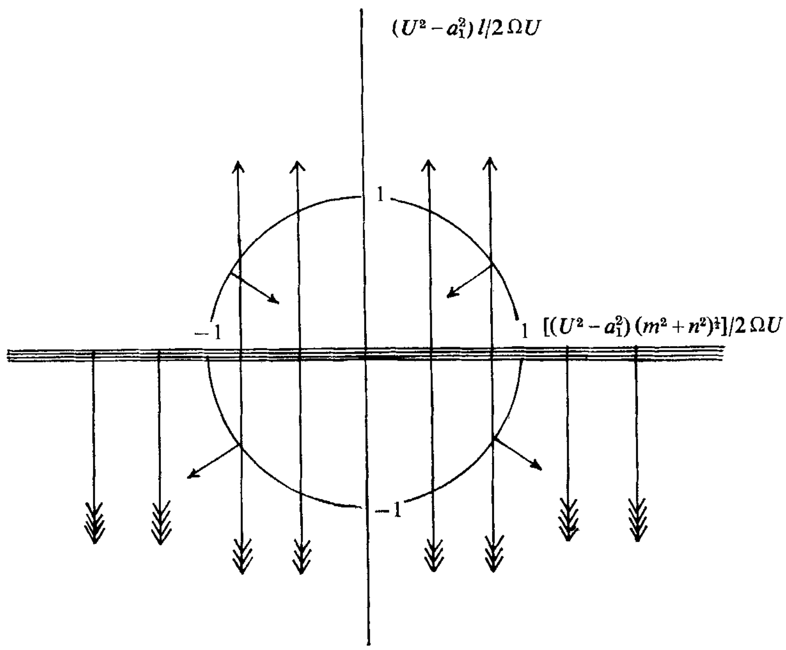

FIGURE 1. $U > a_1$: wave-number surface for inertial waves generated by steady motion of an obstacle with velocity $(U, 0, 0)$ through an infinitely conducting fluid rotating at angular velocity $(\Omega, 0, 0)$ with an applied uniform magnetic field $(H_0, 0, 0)$. It consists of a sphere and four coincident planes.

and otherwise all the four lie below it. The arrows to the planes are shown in figure 1. An analytical discussion is given later. The physical explanation of the result is that these waves with zero phase velocity, whose crests are parallel to the axis of rotation have group velocities

$$C_1 = (1/s)\,(\Omega+(\Omega^2+a_1^2 s^2)^{\frac{1}{2}}), \quad C_2 = (1/s)\,(-\Omega+(\Omega^2+a_1^2 s^2)^{\frac{1}{2}}), \tag{18}$$

where $s = (m^2+n^2)^{\frac{1}{2}}$, directed along the axis either up or down it. The inequality (17) can be written as

$$(U-C_1)\,(U+C_2) < 0, \tag{19}$$

where (19) is satisfied, C_1 exceeds U so that forward influence becomes possible, otherwise U exceeds all the group velocities so that all the disturbances trail behind the obstacle. The waves propagate without attenuation because the associated part of the wave-number surface is plane. After a long enough time the waves satisfying (19) extend arbitrarily for both ahead and behind the obstacle in a Taylor column.

If the Rossby number defined by

$$(U^2-a_1^2)/2\Omega Ua \tag{20}$$

is large where 'a', the transverse dimension of the obstacle, is small, then it cannot significantly excite waves satisfying (17). But as the ratio (20) decreases, transverse disturbances satisfying (17) can be increasingly excited by the obstacle. The transverse disturbance extending ahead of the obstacle is subjected to a 'low pass filter' passing only waves with wave-numbers below $2\Omega U/(U^2-a_1^2)$. The disturbances that extend behind the obstacle are not subjected merely to a complementary 'high pass filter' but include some low wave-number terms also. This result is in contrast with Rossby waves but similar to the case without magnetic field (Lighthill 1967).

Case 2. $U = a_1$

The wave-number surface consists of four coincident planes. There are no spherical waves. The arrows to the planes point one ahead and three behind so that waves propagate without attenuation both ahead and behind the obstacle.

Case 3. $U < a_1$

The wave-number surface consists of the sphere

$$l^2+m^2+n^2 = \left(\frac{2\Omega U}{a_1^2-U_2}\right)^2 \tag{21}$$

and four coincident planes (figure 2). The wave-numbers on this sphere correspond to waves of uniform length $\pi(a_1^2-U^2)/\Omega U$ and arbitrary direction. The direction of the arrows on the sphere are such that these waves are found only ahead of the obstacle. The directions of the arrows to the four planes are obtained in the same way as in the previous case. Three of them point downwards and one upwards when

$$m^2+n^2 < 2\Omega U/(a_1^2-U^2), \tag{22}$$

and otherwise two point upwards and two point downwards. For a physical explanation the condition (22) can be written as

$$(U+C_1)(U-C_2) > 0. \tag{23}$$

From (18) it follows that $C_1 > a_1$ and further $C_1 > a_1 > U$ (from case 3). From (23) $U > C_2$, therefore $C_1 > a_1 > U > C_2$. When (23) is not satisfied both C_1 and C_2 exceed U. This explains the arrows shown in figure 2. Thus in this case the forward influence is possible even when (23) is not satisfied. This result is in contrast with the case $U > a_1$ and the case without magnetic field. The waves propagate without attenuation both ahead and behind the obstacle in a Taylor column without any restriction on the wave-numbers or Rossby number.

To estimate the magnitude of the unattenuated disturbances, the method leading to (14) cannot be used without change because the integral to be estimated has a fourth-order pole singularity on four times covered portions of the wave-number surface. The method is modified as follows: Let us consider

$$\mathbf{V} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp i(my+nz)\,dm\,dn\int_{-\infty}^{\infty}\frac{F(l,m,n)}{G(l,m,n)}\exp il(x-Ut)\,dl, \tag{24}$$

where $$G(l,m,n) = [a_1^2 l^2 - (Ul+i\epsilon)^2]^2 (l^2+m^2+n^2) - 4\Omega^2 (Ul+i\epsilon)^2 l^2. \tag{25}$$

By putting $\epsilon = 0$ in (25) the equation (24) coincides with (9). The problem is to estimate the inner integral of (24) when $|x-Ut|$ is large. When ϵ is positive but very small, the fourth-order pole $l = 0$ is split into four simple poles at

$$l_1 = \frac{-is\epsilon}{-\Omega+Us-(\Omega^2+a_1^2 s^2)^{\frac{1}{2}}}, \quad l_2 = \frac{-is\epsilon}{-\Omega+Us+(\Omega^2+a_1^2 s^2)^{\frac{1}{2}}},$$
$$l_3 = \frac{-is\epsilon}{\Omega+Us-(\Omega^2+a_1^2 s^2)^{\frac{1}{2}}}, \quad l_4 = \frac{-is\epsilon}{\Omega+Us+(\Omega^2+a_1^2 s^2)^{\frac{1}{2}}}. \tag{26}$$

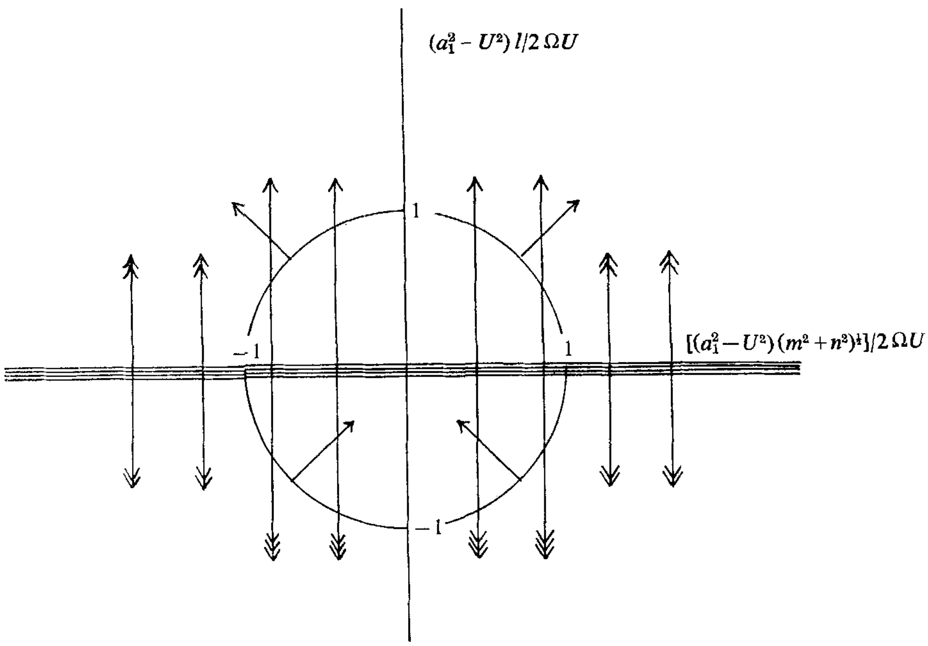

FIGURE 2. $U < a_1$: wave-number surface for inertial waves generated by steady motion of an obstacle with velocity $(U, 0, 0)$, through an infinitely conducting fluid rotating at angular velocity $(\Omega, 0, 0)$ with an applied uniform magnetic field $(H_0, 0, 0)$. It consists of a sphere and four coincident planes.

When (17) is satisfied, $l_1 > 0$ and $l_2, l_3, l_4 < 0$, so that by Jordan's lemma there is a contribution to the integral from the pole $l = l_1$ when $x-Ut > 0$ and from $l = l_2, l_3, l_4$ when $x-Ut < 0$. But when (17) is not satisfied all l_1, l_2, l_3, l_4 have negative imaginary parts and there is no contribution at all for $x-Ut > 0$. This agrees with the direction of the arrows in figure 1.

When (22) is satisfied, there is a contribution from l_1 when $x-Ut > 0$ and from l_2, l_3, l_4 when $x-Ut < 0$. But when (22) is not satisfied l_1 and l_3 contribute for $x-Ut > 0$ and l_2, l_4 contribute for $x-Ut < 0$. This agrees with the direction of the arrows in figure 2.

The contribution to the inner integral of (24) by the residue at the pole $l = l_1$ is

$$\frac{-\pi i F(l_1,m,n)\exp[il_1(x-Ut)]}{[(a_1^2-U^2)^2-4\Omega^2U^2](l_1-l_2)(l_1-l_3)(l_1-l_4)}. \tag{27}$$

Similarly, the contributions due to the poles $l = l_2, l_3, l_4$ can be written down. The difficulty in taking the limit as $\epsilon \to 0$ disappears when one realises that for a steady disturbance (8) reduces to

$$\mathbf{f} = (\partial^3/\partial x^3)\,(U^3 \operatorname{curl} \mathbf{g} - U a_1^2 \operatorname{curl} \mathbf{g} - 2\Omega U^2 \mathbf{g}), \tag{28}$$

whose Fourier transform $F(l, m, n)$ contains a factor l^3. Then taking the limit as $\epsilon \to 0$ of (27) and similar expressions for l_2, l_3 and l_4 by L'Hospital's rule, we get

$$\frac{-\pi i\,(\partial F/\partial l)|_{l=0}\, s(-\Omega + Us \pm (\Omega^2 + a_1^2 s^2)^{\frac{1}{2}})}{8\Omega(\Omega^2 + a_1^2 s^2)^{\frac{1}{2}}\,(\Omega \pm (\Omega^2 + a_1^2 s^2)^{\frac{1}{2}})\,[(U^2 - a_1^2)\, s - 2\Omega U]}, \tag{29}$$

$$\frac{-\pi i(\partial F/\partial l)|_{l=0}\, s(\Omega + Us \pm (\Omega^2 + a_1^2 s^2)^{\frac{1}{2}})}{8\Omega(\Omega^2 + a_1^2 s^2)^{\frac{1}{2}}\,(\Omega \pm (\Omega^2 + a_1^2 s^2)^{\frac{1}{2}})\,[(U^2 - a_1^2)\, s + 2\Omega U]}, \tag{30}$$

where the upper signs in (29) and (30) are the limits for $l = l_1$, $l = l_3$ and lower signs are the limits for $l = l_2$ and $l = l_4$ respectively. The expressions (29) and (30) give the estimates of the inner integral of (24).

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REFERENCES

Lighthill, M. J. 1960 *Phil. Trans.* A **252**, 397.
Lighthill, M. J. 1967 *J. Fluid Mech.* **27**, 725.
Nigam, S. D. & Nigam, P. D. 1963 *Proc. Roy. Soc.* A **272**, 529.
Taylor, G. I. 1922 *Proc. Roy. Soc.* A **102**, 180.